

Cosmology-Course @ T8

The isotropic universe

by: Katrin

- Robertson-Walker metric
- Friedmann equations

Literature: Peacock "Cosmological Physics"

K.K. Zuber: "Astroparticle physics"

Lightman et al: "Problem book..."

The Robertson-Walker metric



Cosmology: task of finding
solutions to Einstein's equations
consistent with large-scale
distributions in universe

Relativist

- simplest relativistic generalization of gravity

Modern observer Θ

universe is highly

homogeneous = const. g

isotropic = same in all directions

on large scales

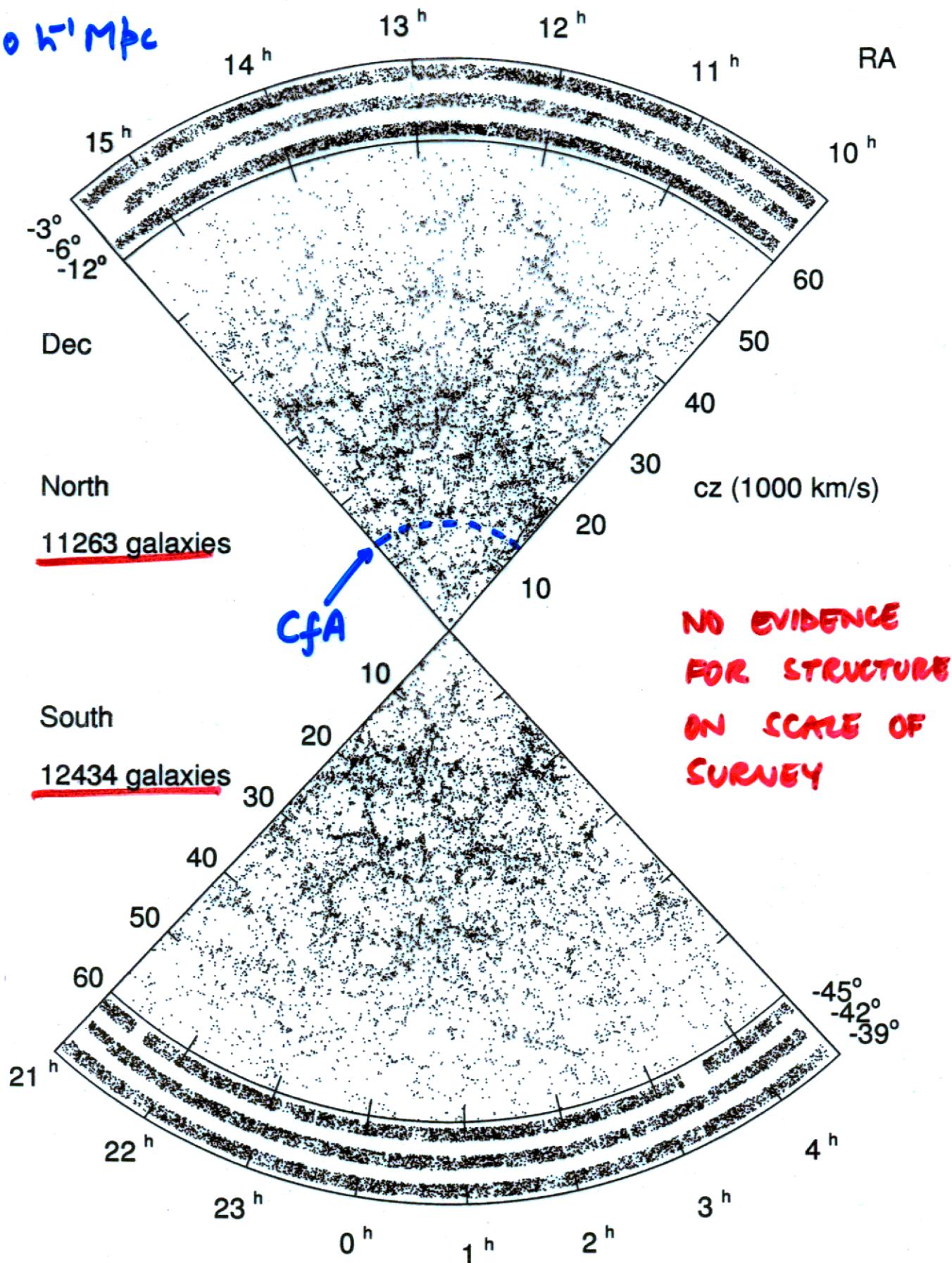
- simplest mass distribution



LAS CAMPAÑAS

15

$\sim 600 h^{-1} \text{Mpc}$

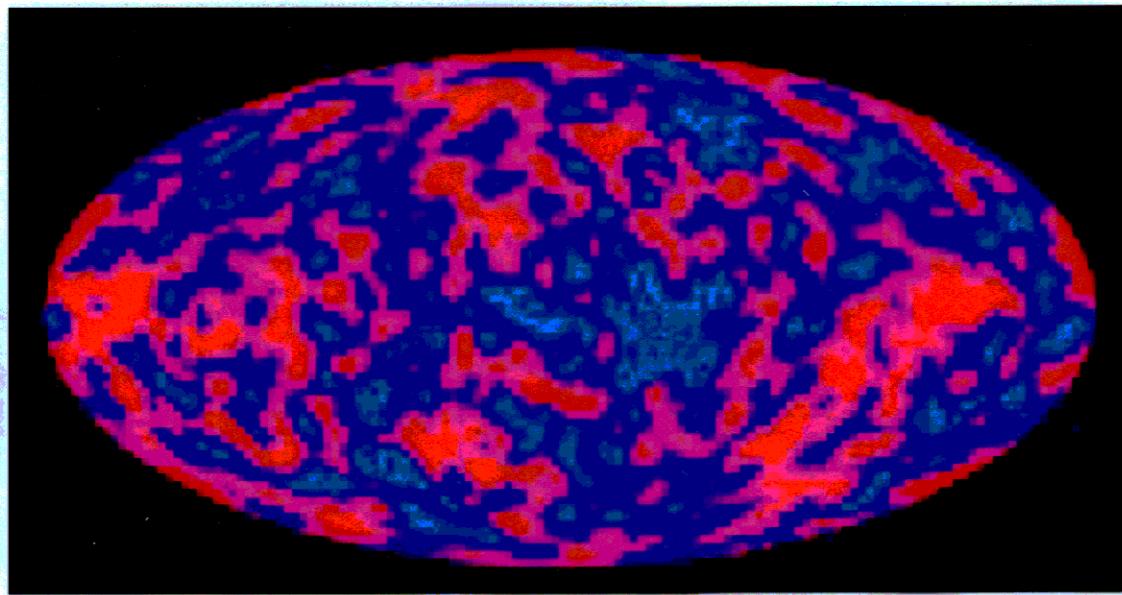


TYPICAL STRUCTURE SCALE $\sim 50 h^{-1} \text{Mpc}$
HOMOGENEITY SPHERE $\sim 100 h^{-1} \text{Mpc}$

The isotropic universe

Katrin Heitmann

- COsmic Background Explorer, 1992



Sky map of the background radiation

scale: $-100 \mu K$ to $100 \mu K$

Isotropy implies homogeneity

- homogeneous $\stackrel{?}{=}$ isotropic NO!
- possible to construct homogeneous universe which is anisotropic
reverse NOT possible!
- consider: \odot surrounded by isotropic m.d.
 $\stackrel{?}{=}$ mass density function of radius only
AND no preferred axis for other physical attributes, i.g. velocity field
 \Rightarrow only allowed velocity field on local scale: expansion or contraction with

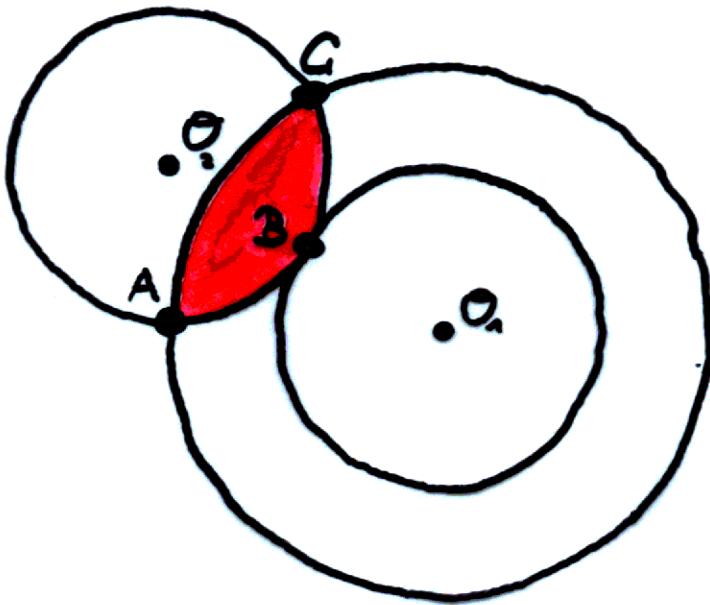
$$\vec{\omega} = H \vec{v}$$

- in 1920's: discovery of isotropic recession above us
What can we learn about universe?
- imagine: explosion \rightarrow debris scattered in all directions with \vec{v}
no self gravity
 \Rightarrow isotropic distribution of objects with recessional velocity \sim distance

Would you see same isotropy from anywhere?

Copernicus: No privileged observer exists!

\Rightarrow if universe isotropic for us, then for every observer in other galaxy
"isotropic about all directions"
 \Rightarrow universe is homogeneous everywhere!



- homogeneous sphere around Θ_1
 - isotropy around Θ_1 and Θ_2
 - \Rightarrow homogeneity at A, B, C
 - \Rightarrow red region homogeneous
 - use large enough shells \Rightarrow extension to whole universe
- isotropy $\xrightarrow{\text{---}} \text{homogeneous}$

- homogeneous but anisotropic:

model, that expands at 3 different rates along 3 axes

\Rightarrow homogeneity preserved

distance-redshift relation could be

function of direction on sky!

- choose model mass distribution

isotropic + homogeneous

- derive metric \rightarrow RW-metric (1936)

- insert metric in Einstein's equations

\rightarrow Friedmann models

- solve equations \rightarrow see what's happening

Derivation of Robertson-Walker metric

different approaches:

- "Problembook", 19.4
calculate R_{ij} , Γ^i_j 's \rightarrow metric
- ... arguments think
- consider: set of fundamental O_s
- define: global time t
 t measured by all O_s , synchronized
by exchange of light signals
set clocks to standard time, e.g.
when universal g reaches some value

$$\Rightarrow t + \text{isotropy}: c^2 ds^2 = c^2 dt^2 \cdot R^2(t) [f^2(r) dr^2 + g^2(r) d\Omega^2]$$

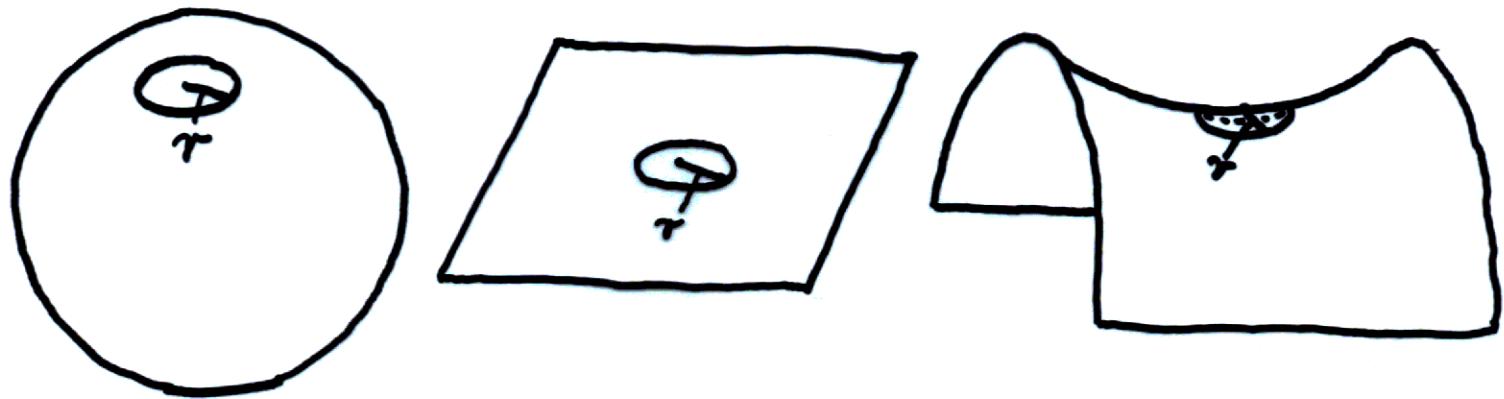
- used: equivalence principle
proper time interval between two events
looks locally like special relativity
to ∂ : $c^2 dz^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2$
same t for them and us
- remaining problem: relating their dx to our
- spherical symmetry: $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$
- distance decomposed in product of
 t -dependent scale-factor $R(t)$
constant, comoving coordinates τ
- f.g.: arbitrary, $f=1$ or $g=r^2 \rightarrow$ Euclidian
- form of remaining function: symmetry

- Simpler case: metric on surface of sphere
"expanding universe $\hat{=}$ balloon being inflated"
- length element on surface of sphere with radius R : $d\tilde{s}^2 = R^2(d\tau^2 + \sin^2\tau d\phi^2)$
- converting to metric with negative curvature:
 $R \rightarrow iR$, $\tau \rightarrow i\tau$: $d\tilde{s}^2 = R^2(d\tau^2 + \sinh^2\tau d\phi^2)$
- combining both: $d\tilde{s}^2 = R^2 \left[\frac{dt^2}{1-kr^2} + r^2 d\phi^2 \right]$
 - $k=+1$: positive curvature
 - $k=-1$: negative curvature

$\hat{=}$ expanding universe with no center!
- small separations $\hat{=}$ space Euclidean
 - \rightarrow simple scaling up $\vec{x}(t) \sim R(t) \vec{x}(t_0)$
- independent of origin! Everybody in center!
 $\vec{x}_1(t) - \vec{x}_2(t) \sim R(t) [\vec{x}_1(t_0) - \vec{x}_2(t_0)]$

\Rightarrow Robertson-Walker-metric

$$c^2 d\tau^2 = c^2 dt^2 - R^2(t) \left[\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \right]$$



$$k=+1$$

closed

$$k=0$$

flat

$$k=-1$$

open

Conformal time

- distinction between physical time t and comoving time η

$$\eta = \int_0^t \frac{cdt'}{R(t')} \quad d\eta = \frac{cdt}{R(t)}$$

$$\Rightarrow c^2 d\tau^2 = R^2(\eta) \left[d\eta^2 - \frac{dr^2}{1-kr^2} + r^2 d\theta^2 \right]$$

The redshift

- Hubble-law: $v = H_0 r$
↑
today
- light source emits wave at (r_1, t_1)
measurement by us at $(r=0, t_0)$
photon travels on null geodesic $dr^2 = 0$

$$\Rightarrow \text{RW-metric: } \int_{t_1}^{t_0} \frac{dt}{R(t)} = \int_0^{r_1} \frac{dr}{\sqrt{1 - kr^2}} \quad d\phi = d\theta = 0$$

w.l.o.g.

next wave, shortly afterwards δt emitted:

$$\int_{t_1}^{t_0} \frac{dt}{R(t)} = \int_{t_1 + \delta t_1}^{t_0 + \delta t_0} \frac{dt}{R(t)} \quad (\text{source fixed in moving reference frame})$$

δt very small $\rightarrow R = \text{const}$ over integration period

$$\Rightarrow \frac{\delta t_1}{R(t_1)} = \frac{\delta t_0}{R(t_0)}$$

- δt = distance of two sequent waves
 - ⇒ corresponds to λ at emission/absorption
- $\Rightarrow \frac{\lambda_0}{\lambda_1} = \frac{R(t_0)}{R(t_1)} \equiv 1 + z$
 - \uparrow
red-shift
- z connected to size of universe at specific t
 - ⇒ observe $z \approx$ information about $R(t)$
- Taylor: $\frac{R(t)}{R(t_0)} = 1 + \underbrace{\frac{\dot{R}(t_0)}{R(t_0)}(t-t_0)}_{H_0} + \frac{1}{2} \underbrace{\frac{\ddot{R}(t_0)}{\dot{R}^2(t_0)} R(t_0) H_0^2 (t-t_0)^2}_{-q_0: \text{deceleration}}$
 - with $\tau = c(t-t_0)$, first term: parameter
 - $\Rightarrow 1+z = 1+H_0 \frac{\tau}{c} \Rightarrow Cz = H_0 \tau$
 - ⇒ shift of λ analogous Doppler-effect
 - but: Doppler-effect: due to relative motion
cosm. redshift: increase of λ due to expansion of space!

Dynamics of the expansion

Expansion and geometry

- "Problem book", 19.13/14

take RW-metric, Einstein's equations

... calculate Christoffel-symbols ...

$$\dots \Gamma_{00}^0 = \frac{R'}{R} \quad \Gamma_{\alpha\beta}^0 = -\frac{R'}{R} g_{\alpha\beta} \quad \Gamma_{0\beta}^\alpha = \frac{R'}{R} \delta_\beta^\alpha$$

$$\Gamma_{\alpha 0}^0 = \Gamma_{00}^\alpha = 0 \quad \dots$$

- Or: model for perfect fluid, symmetries

in metric $\hat{=}$ symmetries in $T_{\mu\nu}$

$$T_{\mu\nu} = \text{diag } (g, -p, -p, -p)$$

$$\Rightarrow 0\text{-comp.: } \left. \frac{\dot{R}^2}{R^2} + \frac{k c^2}{R^2} = \frac{8\pi G}{3} g \right\} \Rightarrow \frac{\ddot{R}}{R} = -\frac{4\pi G}{3} (g + 3p)$$

$$\text{Space: } 2\frac{\ddot{R}}{R} + \frac{\dot{R}^2}{R^2} + \frac{k c^2}{R^2} = -8\pi G p$$

Friedmann-equations

- equations cover all contributions to ρ : matter, radiation, vacuum energy
- connection between $\rho \leftrightarrow$ geometry
- for flat universe: $k = 0$

$$\Rightarrow \rho_c = \frac{3H^2}{8\pi G} \quad \text{"critical density"}$$

$\rho > \rho_c$: spatially closed

$\rho < \rho_c$: spatially open

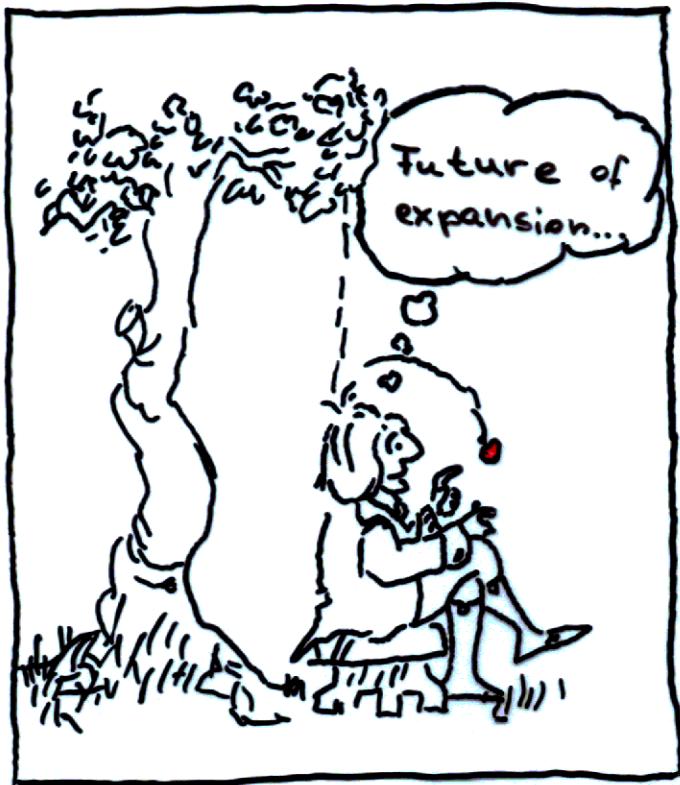
• ρ_c defines density parameter: $\Omega := \frac{\rho}{\rho_c} = \frac{8\pi G \rho}{3H^2}$

ρ, H t-dependent $\rightarrow \Omega$ epoch dependent

• by defining $h_0 = \frac{H_0}{100 \text{ km s}^{-1} \text{ Mpc}^{-1}}$

$$H_0 \approx (65 \pm 10) \text{ km/s Mpc}$$

\Rightarrow current density: $\rho_0 = 1,88 \cdot 10^{-26} \Omega h_0^2 \text{ kg/m}^{-3}$



$$\dot{R} - \frac{8\pi G}{3} g R^2 = -kc^2$$

- future of expansion depends on g
- if universe expands sufficiently fast \rightarrow forever

- if $g > g_c$ gravity holds eventually expansion
- solution of Friedmann-equation shows almost this behavior: for $g_v = 0$ depending on geometry:
 - universe expands forever
 - OR eventually recollapses
 \Rightarrow investigate in more detail!

Solutions of the Friedmann-equations

- restrict to: nonrelativistic dust - .
radiation
vacuum energy

ONLY

$$\dot{R}^2 + c^2 k - \frac{8\pi G}{3} g R^2 = 0 \quad (*)$$

$$\ddot{R} + \frac{4\pi}{3} G (g + 3p) R = 0 \quad (**)$$

$$\frac{d}{dt} (*) = \dot{g} R^3 - \ddot{R} \dot{R} R \frac{3}{4\pi G} + 2 R^2 \dot{R} \dot{g} = 0$$

$$\stackrel{(***)}{\Rightarrow} \dot{g} R^3 + 3 \dot{R} R^2 (g + p) = 0$$

$$\Rightarrow d(gR^3) = -p d(R^3)$$

"First law of thermodynamics"

- take simple equation of state:

$$p = \alpha \rho \quad \alpha = \text{const}$$

$$\Rightarrow g \sim R^{-3(1+\alpha)}$$

- non-relativistic dust: $p=0 \rightarrow g \sim R^{-3}$
- hot, ultrarelativistic gas: $p=\frac{1}{3}g \rightarrow g \sim R^{-4}$
- vacuum energy: $p=-g \rightarrow g = \text{const.}$
- first two cases: $\frac{8\pi G}{3}g > \frac{k}{R^2}$ for R small
 $\Rightarrow \left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi G}{3}g$
 $\Rightarrow \dot{R} \sim R^{-\frac{1}{2}(1+3\alpha)}$
 $\Rightarrow R \sim t^{2/3(1+\alpha)}$
- non-relativistic dust: $R \sim t^{2/3}$ matter dom
- ultra-relativistic gas: $R \sim t^{1/2}$ radiation dom
 \Rightarrow special point, where $g_m = g_{sr}$

• matter density today: $\rho_m = 1,88 \cdot 10^{-28} R_0 h^2 \text{ g cm}^{-3}$

• radiation density of 3K-background:
 $\rho_{so} = 4,8 \cdot 10^{-34} \text{ g cm}^{-3}$

$$\Rightarrow \frac{\rho_{so}}{\rho_m} \approx \frac{R_0}{R} = 1+z \Rightarrow 1+z_{eq} = 2,3 \cdot 10^9 R_0 h^2$$

$\Rightarrow z \approx 1500$ transition radiation \rightarrow matter dominated universe

NOW: Observational friendly!

• density written in observables:

$$\frac{8\pi G}{3} \rho = H_0 (\Omega_r + \Omega_m a^{-3} + \Omega_\Lambda a^{-4}) \quad a = \frac{R(t)}{R_0}$$

$$\frac{8\pi G}{3} \rho = \frac{\dot{R}^2}{R^2} + \frac{k c^2}{R^2} = H^2 + \frac{k c^2}{R^2} \quad \rho = \frac{34^2}{8\pi G} \Omega$$

$$\Rightarrow H^2 (\Omega - 1) = k \frac{c^2}{R^2}$$

$$\Rightarrow \frac{k c^2}{H^2 R^2} = \Omega - 1 = \Omega_m(a) + \Omega_\Lambda(a) + \Omega_\Lambda(a) - 1$$

• for flat universe: $k = 0$

$\Rightarrow \sum_i \Omega_i \equiv \Omega_{\text{tot}} = 1$ independent of
 $\Omega_m, \Omega_r, \Omega_\nu$

• present observation from CMB:

$$\Omega_{\text{tot}} \approx 1.05$$

now: solve Friedmann-equations

• rewrite in conformal time

$$\frac{d}{d\eta} = \frac{R}{c} \frac{d}{dt} \Rightarrow \dot{R} = c \frac{R'}{R}$$

$$\Rightarrow \frac{R'^2}{R_0^2} = \frac{8\pi G}{3c^2} \rho R^4 \frac{R^2}{R_0^4} - k \frac{R^2}{R_0^2} \quad | : R_0^2$$

$$\Rightarrow a'^2 = \frac{R_0^2 H_0^2}{c^2} \Omega a^4 - k a^2 \quad / \frac{R_0^2 H_0^2}{c^2} = \frac{k}{1-\Omega}$$

$$\Rightarrow a' = \frac{k}{\Omega-1} \left[\Omega \frac{H^2}{H_0^2} a^4 - a^2 (\Omega - 1) \right]$$

$$\text{remember: } H^2 \Omega = H_0^2 (\Omega_\nu + \Omega_m a^{-3} + \Omega_r a^{-4})$$

$$\Rightarrow a'^2 = \frac{k}{\Omega - 1} [\Omega_0 + \Omega_m + \Omega_r - a^2(\Omega - 1)]$$

- for $\Omega_0 = 0 \rightarrow$ straightforward to integrate
- for observations: connection to H

$$H = \frac{\dot{R}}{R} = \frac{R'c}{R^2} = \frac{R'}{R_0} c \frac{R_0^2}{R^2} \frac{1}{R_0} = \frac{a'}{a^2} \frac{c}{R_0}$$

$$\Rightarrow \frac{a'^2}{a^4} \frac{c^2}{R_0^2} = H^2(a) = \frac{c^2}{R_0^2} \frac{R_0^2 H_0^2}{c^2} [\dots]$$

$$\Rightarrow H^2(a) = H_0^2 [\Omega_0 + a^{-3} \Omega_m + a^{-4} \Omega_r - a^{-2} (\Omega - 1)]$$

\Rightarrow relation between z and comoving distance

• radial equation of motion for photon:

$$R dr = c dt = c \frac{dR}{\dot{R}} = c \frac{dR}{RH}$$

$$\Rightarrow \frac{R_0}{1+z} dr = c \frac{dR}{RH}$$

$$\Rightarrow R_0 dr = (1+z) c \frac{dR}{RH} = \underbrace{(1+z)}_{a^{-1}} c \underbrace{\frac{R}{R_0}}_{\frac{a}{H}} \frac{da}{H} = \frac{c}{H} dz$$

$$\Rightarrow R_0 dz = \frac{c}{H_0} [\Omega_0 + (1+z)^3 \Omega_m + (1+z)^4 \Omega_0 + (1-\Omega)(1+z)^2]^{1/2} dz$$

Relation between: comoving distance,
red-shift, Hubble-constant,
density parameter!

• red-shift dependence of Ω :

$$\Omega^{-1} = \frac{\Omega-1}{a^2 \Omega_0 + a^{-1} \Omega_m + a^{-2} \Omega_r - \Omega + 1}$$

\Rightarrow at high redshift: z large $\rightarrow a^{-1}$ large

all universes (apart from those with
only Ω_0) tend to look like

$\Omega = 1$ -models!

\Rightarrow fine-tuning of density and expansion rate
= flatness problem

Models with vacuum energy

- non-zero cosmological constant

$$\Omega_0 = \frac{8\pi G}{3} \rho_0 = \frac{\Lambda c^2}{3H^2}$$

- first case: only ρ_0 "de Sitter-model"

$$\ddot{R}^2 - \frac{8\pi G}{3} \rho R^2 = -k c^2$$

$\rho = \text{const} \rightarrow R \text{ increases without limit}$

T. h. s. negligible, set $k=0$

$$\Rightarrow R \sim e^{Ht} \quad H^2 = \frac{\Lambda c^2}{3}$$

- interpretation: Λ caused expansion!

with $\Lambda=0$: expansion from init. cond.

now: vacuum repulsion provides mechanism
to set expansion into motion

\Rightarrow INFLATIONARY COSMOLOGY

General case

- equation of state for vacuum:

$$\rho_v = -g_v \quad \text{perfect fluid}$$

$$\Rightarrow \left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi G}{3} (\rho + \rho_v) - \frac{kc^2}{R^2}$$

$$\frac{\ddot{R}}{R} = -\frac{4\pi G}{3} (\rho - 2\rho_v + 3p)$$

- neglect radiation \rightarrow dust $p=0$

otherwise: **NUMERICS!**

(1) static solution: $\dot{R} = \ddot{R} = 0$

$$\Rightarrow \rho = 2\rho_v \quad (*) \quad \frac{8\pi G}{3} R^2 (\rho + \rho_v) = kc^2 \quad (1)$$

$$(*) \sim \rho_v > 0 \quad \sim (***) \quad k=1 \Rightarrow R^2 = \frac{1}{4\pi G \rho}$$

attractive force of ρ $\hat{=}$ repulsion of $-1 > 0$

\Rightarrow static, closed universe (... NEWTON)

but: INSTABLE!

$$R^2 = \frac{1}{4\pi G g}$$

increase $R \rightarrow$ decrease g

Λ still constant!

\rightarrow repulsion dominant

\Rightarrow solution drifts away from static case!

(2) non-static solutions

$\Lambda > 0$: always acceleration of expansion

$\Lambda < 0$: always deceleration of expansion

from $\frac{\dot{R}^2}{R^2} = \frac{8\pi G}{3}(g + g_v) - \frac{k}{R^2}$

for R large $\rightarrow \Lambda$ dominant, since $g_v = \text{const}$

$\rightarrow \Lambda < 0$ collapsing universe

$\Lambda > 0, k = -1, 0 \Rightarrow$ always positive

solution, expands forever

$k = 1 \Rightarrow \Lambda_c$ Einstein's static solution

- $0 < \Lambda_c < \Lambda$: solution without initial singularity
- $\Lambda = \Lambda_c(1+\epsilon)$, $\epsilon \ll 1$: Lemaitre - universe
with almost static phase

